

Light bending and perihelion precession: A unified approach

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(February 5, 2008)

Abstract

The standard General Relativity results for precession of particle orbits and for bending of null rays are derived as special cases of perturbation of a quantity that is conserved in Newtonian physics, the Runge-Lenz vector. First this method is applied to give a derivation of these General Relativity effects for the case of the spherically symmetric Schwarzschild geometry. Then the lowest order correction due to an angular momentum of the central body is considered. The results obtained are well known, but the method used is rather more efficient than that found in the standard texts, and it provides a good occasion to use the Runge-Lenz vector beyond its standard applications in Newtonian physics.

Pacs: 04.20.-q, 04.70.Bw, 04.25.-g

Typeset using REVTeX

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I. INTRODUCTION

Light bending and perihelion precession are the two most important effects on orbits caused by the General Relativity corrections to the Newtonian gravitational field of the sun. The standard derivation treats these two effects in different ways, without any apparent connection between them. Yet, in the usual Schwarzschild coordinates they are both due to the same, single relativistic correction to the Newtonian potential, so it is of some interest to use the same method to derive either effect.

The key to the present unified treatment is the Runge-Lenz vector. In Newtonian physics, where the two effects are absent, this vector is constant and points from the center of attraction to the orbit's perihelion.¹ Its non-constancy in General Relativity therefore is a measure of either effect. The Runge-Lenz vector was established as a useful tool by 1924 at the latest, but it did not become popular until the 1960's.²

Since then a number of papers that exploit its advantages have graced the pages of this Journal (see ref. [2] and the references cited therein), and the results to be reported here can in essence be found in earlier papers, but the unified viewpoint vis a vis General Relativity is perhaps new. In addition the “magnetic” gravitational effects due to a rotating central body are treated here with this method.

II. GENERAL RELATIVISTIC EQUATIONS OF MOTION (NO ROTATION)

The motion to be considered is that of a “test particle” that moves along a geodesic in the spacetime exterior to the central body. If this body is non-rotating, spherically symmetric, and has total mass M , the exterior spacetime geometry is described by the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2. \quad (2.1)$$

Here the coordinates t, r, θ and ϕ are one of many equally valid choices for labeling spacetime points, but they can nevertheless be invariantly characterized [3]. (For example, $4\pi r^2$ is the area of the sphere $r = \text{const}$, $t = \text{const}$, and $\partial/\partial t$ is a timelike Killing vector.) The geodesic law of motion is essentially equivalent to the conservation laws that follow from the symmetries of the geometry and the conservation of rest mass. Because of the principle of equivalence we may assume without loss of generality that the rest mass is ϵ , where $\epsilon = 1$ for particles of finite rest mass, and $\epsilon = 0$ for light (photons). The other conserved quantities correspond to angular momentum \mathbf{L} , and energy E . We use τ to denote the proper time

¹Although our treatment is not confined to the solar system, we use this term to denote the point of closest approach to the center because it seems more familiar (and more etymologically consistent) than the more correct term, pericenter.

²For a history of the Laplace-Runge-Lenz vector, see ref. [1].

along the geodesic.³ As usual we can choose the vector \mathbf{L} to be normal to the plane $\theta = \pi/2$ and we then have⁴

$$E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \quad (2.2)$$

$$L = r^2 \frac{d\phi}{d\tau} \quad (2.3)$$

$$\mathcal{E} \equiv \frac{1}{2}(E^2 - \epsilon) = \frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 - \frac{\epsilon M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}. \quad (2.4)$$

Equation (2.4) has the form of conservation of energy in an effective potential. Except for the presence of τ instead of t , Eqs (2.2 - 2.4) are the same as the Newtonian equations of motion of a particle of unit mass and total energy \mathcal{E} in a potential $V = -\epsilon M/r - ML^2/r^3$. Thus for particles as well as for light the relativistic motion in proper time τ is the same as the Newtonian motion in Newtonian time t if the potential is modified by the single term $-ML^2/r^3$. We note that no slow motion assumption or other approximation is involved in this correspondence. If we are only interested in the orbit equation, then the difference between t and τ does not matter, because either one will be eliminated in the same way in favor of ϕ via Eq (2.3).

III. SECULAR CHANGE OF ORBITS

We treat the modification ML^2/r^3 of the Newtonian potential as a perturbation⁵ and compute the consequent changes in direction of the Runge-Lenz vector, defined by

$$\mathbf{A} = \mathbf{v} \times \mathbf{L} - \epsilon M \mathbf{e}_r \quad (3.1)$$

where all bold-face quantities are 3-vectors, and \mathbf{e}_r denotes a unit vector in the \mathbf{r} -direction. The time parameter is τ as above, so that, for example, $\mathbf{v} = d\mathbf{r}/d\tau$. The rate of change of \mathbf{A} is (as, for example, in ref. [5])

$$\frac{d\mathbf{A}}{d\tau} = \left(r^2 \frac{\partial V}{\partial r} - \epsilon M\right) \frac{d\mathbf{e}_r}{d\tau} = \left(\frac{3ML^2}{r^2}\right) \frac{d\phi}{d\tau} \mathbf{e}_\phi. \quad (3.2)$$

The direction of \mathbf{A} therefore changes with angular velocity

$$\boldsymbol{\omega} = \frac{\mathbf{A} \times \dot{\mathbf{A}}}{\mathbf{A}^2} = \left(\frac{3ML^2}{A^2 r^2}\right) \frac{d\phi}{d\tau} \mathbf{A} \times \mathbf{e}_\phi \quad (3.3)$$

³For the case of light, τ is an affine parameter that is defined only up to scale transformations. The quantities E and L are therefore similarly defined only up to such re-scaling. The final, physical results will contain only ratios of such quantities.

⁴For a derivation, see references [3,4] or section IV below.

⁵This means that $M/r \ll 1$, where r is a typical orbit radius, which follows if we assume $M/L \ll 1$

and its total change when the particle moves from ϕ_1 to ϕ_2 is (assuming this change is small, and that \mathbf{A} is originally in the $\phi = 0$ direction)

$$\Delta\alpha = \int_{\phi_1}^{\phi_2} \omega d\tau = 3ML^2 \int_{\phi_1}^{\phi_2} \frac{\cos \phi d\phi}{Ar^2}. \quad (3.4)$$

When \mathbf{A} is constant, and in the direction $\phi = 0$, we have

$$\mathbf{A} \cdot \mathbf{r} = Ar \cos \phi = L^2 - \epsilon Mr. \quad (3.5)$$

The bound orbits ($\epsilon = 1$) are therefore ellipses with eccentricity $e = A/M$ and semi-major axis $a = \frac{L^2}{M(1-e^2)}$. For the unbound, straight orbits of light rays ($\epsilon = 0$), $L^2/A \equiv b$ is the impact parameter, and because these orbits are traversed at the speed of light we have $L/E = b$. In either case \mathbf{A} points from the center to the perihelion. When \mathbf{A} changes slowly the orbits still have this approximate shape, but their orientation and shape will change slowly. We calculate the lowest order changes due to the General Relativistic correction in V_{eff} by substituting the unperturbed orbit (3.5) into Eq (3.4):

$$\Delta\alpha = \frac{3M}{AL^2} \int_{\phi_1}^{\phi_2} (A \cos \phi + \epsilon M)^2 \cos \phi d\phi. \quad (3.6)$$

A. Perihelion precession

For a particle in a bound orbit it is customary to find the angular change of the perihelion during one revolution (in ϕ) of the particle. Because \mathbf{A} points to the perihelion, this angle is given by Eq (3.6), when $\phi_2 - \phi_1 = 2\pi$.

$$\Delta\alpha = \frac{3M}{L^2} \int_0^{2\pi} \frac{(A \cos \phi + M)^2}{A} \cos \phi d\phi = \frac{6\pi M^2}{L^2} = \frac{6\pi M}{a(1-e^2)}. \quad (3.7)$$

This is the usual perihelion formula.

B. Light bending

Here also the deflection is given by $\Delta\alpha$ of Eq (3.6), but ϕ changes from $-\pi/2$ to $\pi/2$ with respect to the perihelion (and of course $\epsilon = 0$):

$$\Delta\alpha = \frac{3M}{L^2} \int_{-\pi/2}^{\pi/2} A \cos^3 \phi d\phi = \frac{4MA}{L^2} = \frac{4M}{b}. \quad (3.8)$$

This is the usual light deflection formula.

That the light deflection should follow from the same, $O(1/r^3)$ correction to the Newtonian effective potential as the perihelion rotation may be somewhat surprising, because the deflection is frequently heuristically explained as an action of the Newtonian $O(1/r)$ potential on light. Indeed, an effective potential for dr/dt would contain $O(1/r)$ terms, but in the present choice of variables these are absent — illustrating once again the arbitrary nature of coordinates in a generally covariant theory.

IV. SLOWLY ROTATING CENTRAL BODIES

If the body is slowly rotating⁶ in the ϕ -direction with angular momentum J , the metric (2.1) is modified by the Lense-Thirring term [6], $-\frac{4J}{r}\sin^2\theta d\phi dt$ and by a quadrupole term that describes the distortion of the body. The quadrupole term can be treated in the same way as the relativistic correction, and will therefore not be further considered. The Lense-Thirring term breaks the spherical symmetry, so on symmetry grounds only p_ϕ , the angular momentum conjugate to ϕ (called L below), is conserved. Nonetheless the “total angular momentum” $Q^2 = p_\theta^2 + \cot^2\theta p_\phi^2$, where p_θ is the momentum conjugate to θ , is also conserved to first order in J . Thus the entire motion can be formulated in terms of conserved quantities, and one finds that the Newtonian angular momentum \mathbf{L} precesses around the z -direction.⁷ However, we will confine attention to the case when \mathbf{J} and \mathbf{L} are parallel, the motion is confined to the equatorial plane, and the general relativistic correction is completely described by the behavior of \mathbf{A} .

A. Equations of Motion and Runge-Lenz vector

A Lagrangian \mathcal{L} for a particle or a light beam moving in the equatorial plane of metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2 - \frac{4J}{r}\sin^2\theta d\phi dt \quad (4.1)$$

is given by

$$2\mathcal{L} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2M}{r}} + r^2\dot{\phi}^2 - \frac{4J}{r}\dot{\phi}\dot{t}. \quad (4.2)$$

Here the dot indicates differentiation with respect to proper time; this implies that \mathcal{L} is constant,

$$2\mathcal{L} = -\epsilon. \quad (4.3)$$

Since \mathcal{L} is independent of t and ϕ we have the conserved quantities,

$$-E = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -\left(1 - \frac{2M}{r}\right)\dot{t} - \frac{2J}{r}\dot{\phi} \quad L = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\dot{\phi} - \frac{2J}{r}\dot{t}. \quad (4.4)$$

⁶We work only to first order in J ; more precisely, we assume $J \lesssim ML$ and, as before, $M/r \sim M^2/L^2 \sim \epsilon \ll 1$, so that $J/r^2 \sim \epsilon^{3/2}$.

⁷The integration of the equations of motion for orbits of general orientation was one of the aims of the Lense and Thirring paper [6]. For a treatment using the Runge-Lenz vector, see references [7]. For a summary of all the relativistic effects on orbits see reference [8]. For the geometrical reason for the conservation of Q^2 see reference [9].

We solve for \dot{t} and $\dot{\phi}$ to first order in J ,

$$\dot{\phi} = \frac{L}{r^2} + \frac{2JE}{r^3} \quad \dot{t} = \frac{E}{1 - \frac{2M}{r}} - \frac{2JL}{r^3} \quad (4.5)$$

and substitute into Eq (4.3) to obtain a “conservation of energy” in an effective potential,

$$E^2 - \epsilon = \dot{r}^2 - \frac{2\epsilon M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3} + \frac{4JLE}{r^3}. \quad (4.6)$$

The effective potential in this equation contains two non-Newtonian terms. The first was already encountered in Eq (2.4) and causes the “standard” relativistic correction; the second is due to the Lense-Thirring addition to the metric.⁸

Because in the rotating case there is a difference between kinematic angular momentum ($r^2\dot{\phi}$) and canonical angular momentum L , the Runge-Lenz vector \mathbf{A} can be defined in various ways, but there is no difference in the precession rate one calculates from them. A convenient choice is

$$\mathbf{A} = \mathbf{v} \times \left(\mathbf{L} - \frac{2\mathbf{J}E}{r} \right) - \epsilon M \mathbf{e}_r = \left(\frac{L^2}{r} - \epsilon M \right) \mathbf{e}_r - \dot{r} \left(L - \frac{2JE}{r} \right) \mathbf{e}_\phi \quad (4.7)$$

because it simplifies the equation of motion for \mathbf{A} , and still gives elliptical orbits for any J when \mathbf{A} is constant,

$$\mathbf{A} \cdot \mathbf{e}_r = A \cos \phi = \frac{L^2}{r} - \epsilon M. \quad (4.8)$$

The equation of motion for \mathbf{A} can be derived from Eqs (4.5, 4.6):

$$\dot{\mathbf{A}} = \left(\frac{3ML^2}{r^2} - \frac{8\epsilon MJE}{Lr} + \frac{2JE(\epsilon - E^2)}{L} \right) \dot{\phi} \mathbf{e}_\phi. \quad (4.9)$$

By substituting Eq (4.9) into Eq (3.3) and integrating, using (4.8) for $1/r$, we now find that the total change in \mathbf{A} when the particle moves from ϕ_1 to ϕ_2 is

$$\Delta\alpha = \int_{\phi_1}^{\phi_2} \left(\frac{3M}{AL^2} (A \cos \phi + \epsilon M)^2 - \frac{8\epsilon MJE}{AL^3} (A \cos \phi + \epsilon M) + \frac{2JE(\epsilon - E^2)}{AL} \right) \cos \phi d\phi. \quad (4.10)$$

⁸With our assumptions as spelled out in footnote 6 the Newtonian terms of the effective potential are of order ϵ , both non-Newtonian terms are of order ϵ^2 , and typical terms that are neglected are $J^2 E^2 / r^4 \sim JEM L / r^4 \sim \epsilon^3$, $J^2 L^2 / r^6 \sim \epsilon^4$ etc.

B. Perihelion motion

To obtain the perihelion motion we evaluate Equation (4.10) over one revolution ($\phi_1 = 0, \phi_2 = 2\pi$) with $\epsilon = 1$. Since the particle-velocity is nonrelativistic, we may set $E = 1$ to the lowest order:⁹

$$\Delta\alpha = \frac{6\pi M^2}{L^2} - \frac{8\pi JME}{L^3} = \frac{6\pi M}{a(1-e^2)} - \frac{8\pi J}{M^{1/2}(a(1-e^2))^{3/2}}. \quad (4.11)$$

The first term is the “standard” general relativistic precession already found in Section III, and the second term is due to the rotation of the central body.¹⁰ For nearly circular orbits we can interpret this second term as due to two causes: one is the rotation in ϕ of the “locally non-rotating observer” that makes the Lense-Thirring term of Eq (4.2) disappear at the radius of the particle, an amount $4\pi J/aL$; the other is the “differential rotation” due to the $1/r^3$ fall-off of the second non-Newtonian term in the effective potential of Eq (4.6). This contribution causes precession by an amount $-12\pi J/aL$, in the same way as the first non-Newtonian term causes the “standard” precession.¹¹

C. Light bending

For the effect on light we put $\epsilon = 0$ in Eq (4.10) and integrate from $\phi = -\pi/2$ to $\pi/2$ as in section III B,

$$\Delta\alpha = \frac{3MA}{L^2} \cdot \frac{4}{3} - \frac{2JE^3}{AL} \cdot 2 = \frac{4M}{b} - \frac{4J}{b^2}. \quad (4.12)$$

The effect of the angular momentum J on both precession and bending is negative.¹² This is the same “differential” dragging effect that makes a gyroscope in the equatorial plane precess in the opposite direction to the central body’s rotation [8].

V. CONCLUSIONS

For non-rotating spherically symmetric central masses we have seen that the two important general relativistic corrections to the Newtonian gravitational motion, namely perihelion

⁹Formally, this follows from the $M^2 \ll L^2$ assumption and requiring bound orbits.

¹⁰The last term in Eq (4.11) changes sign if L is antiparallel to J . Both terms are of order ϵ .

¹¹For non-equatorial orbits the first contribution is a precession about \mathbf{J} , whereas the second contribution is a precession about \mathbf{L} , proportional to $\mathbf{J} \cdot \mathbf{L}$.

¹²However, the relative contribution of J to light bending is less: we have $M/b \sim \epsilon$, but $J/b^2 \sim \epsilon^{3/2}$.

precession and light bending, follow from the same correction term in the effective radial potential; and that either effect can be viewed as change in the Runge-Lenz vector associated with the orbit. Because both effects follow by simple evaluation from one formula (Eqs 3.6, 4.10), the effort is only about half of the usual procedure; moreover it gives occasion to review and apply the Runge-Lenz vector. We have shown the extension of this calculation to equatorial orbits of a rotating body; relativistic corrections to parabolic and hyperbolic orbits can similarly be evaluated by this method.

REFERENCES

- [1] H. Goldstein, Am. J. Phys. **43**, 735 (1975) and **44**, 1123 (1976)
- [2] C. E. Aguiar and M. F. Banoso, Am. J. Phys. **64**, 1042 (1996)
- [3] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [4] Bernard F. Schutz, *A first course in general relativity* (Cambridge University Press 1985).
- [5] H. Goldstein, *Classical Mechanics* second edition (Addison-Wesley 1980)
- [6] J. Lense and H. Thirring, Phys. Z. **19**, 156 (1918); English translation in Gen. Relativ. Gravit. **16**, 711 (1984). Also see D. R. Brill and J. M. Cohen, Phys. Rev. **143**, 1011 (1966).
- [7] L. D. Landau and E. M. Lifshitz, *The classical Theory of Fields* (Pergamon Press 1975) p. 336; S. Weinberg, *Gravitation and Cosmology* (Wiley 1972) p. 230.
- [8] I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University Press 1995).
- [9] D. Boccaletti and G. Pucacco, Nuovo Cimento **112 B**, 181 (1997).